The evaluation of the far-field integral in the Green's function representation for steady Oseen flow

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(Received 24 May 2006; accepted 22 September 2006)

Consider the Green's function representation of an exterior problem in steady Oseen flow. The far-field integral in the formulation is shown to be zero. © *2006 American Institute of Physics*.
[DOI: 10.1063/1.2388248]

I. INTRODUCTION

Oseen[1] gives the Green's function representation of the exterior problem in steady Oseen flow, but assumes that the far-field integral in the formulation is zero without proof. It is essential to show that this integral is zero for the Oseen representation to be valid. In low Reynolds number flow applications, the Oseen equations are used within singular perturbation theory as a far-field matching to Stokes flow[2,3]. In this case, only the singular point solution is required and so the integral is satisfied trivially. However, there are at least two increasingly important applications of the Oseen equations for high Reynolds number (in the sense that the Reynolds number is much greater than one) flows.

The first application is the decay of the trailing vortex wake behind an aircraft. This has attracted significant recent interest with the advent of superheavy class aircraft, such as the Airbus A380, and the stipulation of safe separation distances between aircraft flying through this wake during landing and takeoff. However, the line vortex in inviscid flow has a constant strength and profile. So in order to model vortex decay, viscosity must be modelled which diffuses the vorticity. Batchelor[4] considers far-field Oseen flow to represent the trailing vorticity as the Oseen formulation is a linearization to a uniform stream of the Navier-Stokes equations, and so retains the viscous term. The Batchelor vortex has been the focus of work on stability analysis for the trailing vorticity, a review given by Delbende.[5] Chadwick[6] shows that the horseshoe vortex in Oseen flow, whose arms are trailing line vortices, is equivalent to a spanwise distribution of lift Oseenlets (a lift Oseenlet is the singular point lift solution in Oseen flow). Furthermore, the trailing vortex behind an aircraft has been developed from rollup of the vortex sheet, and even at large distances behind an aircraft the representation by a line vortex is insufficient and instead a distribution is required (see Ref. 7, chapter 13). The requirement for a distribution of singular solutions means that an integral distribution of singular solutions over a surface, as formulated by Oseen, is necessary. In this case, it is then necessary to show that the far-field integral arising from Oseen's representation is zero.

The second application is in the field of slender body theory and related theories. The usual approach is for inner and outer expansions around the boundary layer, with the inner region being purely viscous. However, Chadwick[8] presents a slender body theory in Oseen flow where it is assumed that for a streamlined body satisfying a Kutta condition at the trailing edge or end section, that Oseen flow (the perturbation to a uniform stream) is valid as an outer expansion. In the application to lift on a slender wing, Chadwick[9] shows that the retention of the viscous terms in the formulation are important for the lift calculation and to ensure the wake is regularized (and so is not singular as in the inviscid flow representation). Again, the Oseen representation is given by a distribution of solutions over a Green's integral surface rather than reducing to point solutions, as in the case of low Reynolds number singular perturbation theory.

It is therefore essential to show that the far-field Green's integral arising from Oseen's representation of the Oseen velocity is zero, for the Oseen representation to be valid for both these important problems. One would assume that a likely way to proceed would be to represent the far-field integral surface as the surface of a sphere, and divide this surface into an interior wake surface and exterior surface where appropriate approximations can be made. However, when this is done then it can only be shown that the far-field integral is bounded by a constant. So, the idea is to find an appropriate division of the far-field surface such that the far-field integral tends to zero as the radius of the sphere tends to infinity. In the present paper, this is achieved by dividing the surface of the sphere into three surfaces by: the intersection of a cone subtended by a small angle and enclosing the wake; and also by the intersection of the wake boundary. Making appropriate approximations within the three regions then enables us to show that the far-field integral in the Green's function formulation of steady Oseen flow is indeed zero as expected.

II. THE OSEEN FORMULATION

The steady Oseen equations (see Ref. 1, pp. 30-38) for the Oseen velocity $\mathbf{u}$, a perturbation to the uniform stream velocity U in the x_1 direction such that the Cartesian coordinates are given by (x_1,x_2,x_3), and Oseen pressure p are

$$\rho U\frac{\partial \mathbf{u}}{\partial x_1} = -\nabla p + (\mu\nabla^2)\mathbf{u}, \quad \nabla\cdot\mathbf{u}=0, \tag{1}$$

$$\nabla^2 p = 0, \tag{2}$$

where ρ and μ are the fluid density and dynamical coefficient of viscosity, respectively, and both are assumed to be constant. ∇ denotes the gradient operator and ∇^2 is the Laplacian operator. As $r\to\infty$, then $\mathbf{u}$, $p\to 0$. The Oseen velocity is then represented by an integral distribution of Green's functions called Oseenlets or Oseen fundamental solutions.[1] Consider four solutions to the Oseen equations $(\mathbf{u},p)$ and $(\mathbf{u}^{(m)},p^{(m)})$, $1\le m\le 3$, for the Oseen velocity and pressure, respectively. From (1) we find that

$$\frac{\partial}{\partial y_1}\{\rho U u_i(\mathbf{y})u_i^{(m)}(\mathbf{x}-\mathbf{y})\}$$
$$=-\frac{\partial}{\partial y_i}\{p(\mathbf{y})u_i^{(m)}(\mathbf{x}-\mathbf{y})+u_i(\mathbf{y})p^{(m)}(\mathbf{x}-\mathbf{y})\}+\mu\frac{\partial}{\partial y_j}$$
$$\times\left\{\frac{\partial u_i}{\partial y_j}(\mathbf{y})u_i^{(m)}(\mathbf{x}-\mathbf{y})-u_i(\mathbf{y})\frac{\partial}{\partial y_j}u_i^{(m)}(\mathbf{x}-\mathbf{y})\right\} \quad (3)$$

for a point $\mathbf{y}=\mathbf{x}$ in the fluid. Applying Gauss's theorem to the volume integral of the above expression gives

$$\int\int_{S_y}\Bigg[p(\mathbf{y})u_j^{(m)}(\mathbf{x}-\mathbf{y})+u_j(\mathbf{y})p^{(m)}(\mathbf{x}-\mathbf{y})$$
$$+\mu\left\{u_i(\mathbf{y})\frac{\partial}{\partial y_j}u_i^{(m)}(\mathbf{x}-\mathbf{y})-\frac{\partial}{\partial y_j}u_i(\mathbf{y})u_i^{(m)}(\mathbf{x}-\mathbf{y})\right\}$$
$$+\rho U u_i(\mathbf{y})u_i^{(m)}(\mathbf{x}-\mathbf{y})\delta_{j1}\Bigg]n_j\mathrm{d}s=0^{(m)}, \quad (4)$$

where S_y is a surface enclosing a volume of fluid, and the integration is over the $\mathbf{y}$ variable. The Green's functions can be represented by the potentials ϕ and χ such that

$$u_i^{(m)}(\mathbf{z})=\frac{\partial\phi^{(m)}}{\partial z_i}+\frac{\partial\chi^{(m)}}{\partial z_i}-2k\chi^*\delta_{mi},$$
$$p^{(m)}(\mathbf{z})=-\rho U\frac{\partial\phi^{(m)}}{\partial z_1}, \quad (5)$$

where $k=\rho U/2\mu$ and $\mathbf{z}=\mathbf{x}-\mathbf{y}$, and

$$\phi^{(m)}(\mathbf{z})=\frac{1}{4\pi\rho U}\frac{\partial}{\partial z_m}\ln(R-z_1),$$
$$\chi^{(m)}(\mathbf{z})=-\frac{1}{4\pi\rho U}e^{-k(R-z_1)}\frac{\partial}{\partial z_m}\ln(R-z_1), \quad (6)$$
$$\chi^*(\mathbf{z})=\frac{1}{4\pi\rho U}\frac{e^{-k(R-z_1)}}{R},$$

where $|\mathbf{z}|=R$. So from (6),

$$\frac{\partial\chi^*}{\partial z_m}=\frac{\partial\chi^{(m)}}{\partial z_1}.$$

Substitute the Green's functions into (4) such that S_y consists of three surfaces: S_0, which encloses a body surface S_B, S_δ, which is a sphere radius $\delta\to 0$ about the point $\mathbf{z}=\mathbf{x}$, and S_R, which is a sphere radius $R\to\infty$ (Fig. 1).

Following[1] the contribution from the surface S_δ is $u_m(\mathbf{x})$, and if the contribution from the surface S_R is assumed to be zero, then we get the Green's function integral representation in Oseen flow:

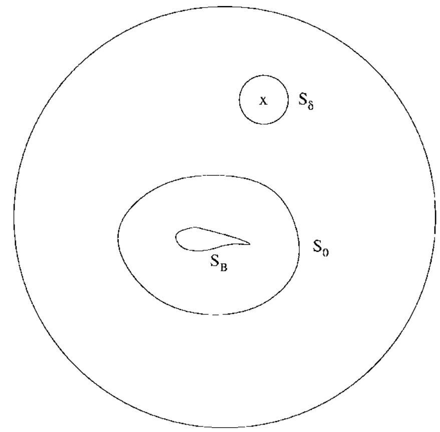

FIG. 1. The division of the surface of the sphere S_R.

$$u_m(\mathbf{x})=-\int\int_{S_0}\Bigg[p(\mathbf{y})u_j^{(m)}(\mathbf{x}-\mathbf{y})+u_j(\mathbf{y})p^{(m)}(\mathbf{x}-\mathbf{y})$$
$$+\mu\left\{u_i(\mathbf{y})\frac{\partial}{\partial y_j}u_i^{(m)}(\mathbf{x}-\mathbf{y})-\frac{\partial}{\partial y_j}u_i(\mathbf{y})u_i^{(m)}(\mathbf{x}-\mathbf{y})\right\}$$
$$+\rho U u_i(\mathbf{y})u_i^{(m)}(\mathbf{x}-\mathbf{y})\delta_{j1}\Bigg]n_j\mathrm{d}s. \quad (7)$$

III. EVALUATION OF THE FAR-FIELD INTEGRAL

The integration over the surface S_R is given by

$$\int\int_{S_R}\Bigg[p(\mathbf{y})u_j^{(m)}(\mathbf{z})+u_j(\mathbf{y})p^{(m)}(\mathbf{z})$$
$$+\mu\left\{u_i(\mathbf{y})\frac{\partial}{\partial y_j}u_i^{(m)}(\mathbf{z})-\frac{\partial}{\partial y_j}u_i(\mathbf{y})u_i^{(m)}(\mathbf{z})\right\}$$
$$+\rho U u_i(\mathbf{y})u_i^{(m)}(\mathbf{z})\delta_{j1}\Bigg]n_j\mathrm{d}s. \quad (8)$$

The surface S_R is such that $|\mathbf{z}|=R$, and we want to show that the integration over this surface tends to zero.

Taking the modulus of (8) and bringing this modulus into the integrand, then we can show that (8) tends to zero if

$$\lim_{R\to\infty}\left\{|u_j(\mathbf{y})|_{max}\int\int_{S_R}|u_i^{(m)}(\mathbf{y})|\mathrm{d}s\right\}=0_{ij}^{(m)} \quad (9)$$

since

$$\left|\frac{\partial u_i^{(m)}(\mathbf{z})}{\partial y_j}\right|\le A_j|u_i^{(m)}(\mathbf{z})|$$

for some constant A_j, and since $|p^{(m)}(\mathbf{z})|\le 1/4\pi R^2$, and $u_i(\mathbf{y})\to 0$ as $R\to\infty$. (In (9), we define $0_{ij}^{(m)}=0$ for all $1\le i,j,m\le 3$.)

To evaluate (9), the integration surface is divided into three (see Fig. 2):

1. The surface S_{wake} such that $|\mathbf{z}|=R$ and $r=\sqrt{z_2^2+z_3^2}\le a_0\sqrt{z_1/k}$, $0<a_0\ll 1$;
2. the surface $S_{\mathrm{cone-wake}}$ such that $|\mathbf{z}|=R$ and $a_0/\sqrt{kz_1}\le\alpha\le\alpha_0$, $0<\alpha_0\ll 1$;

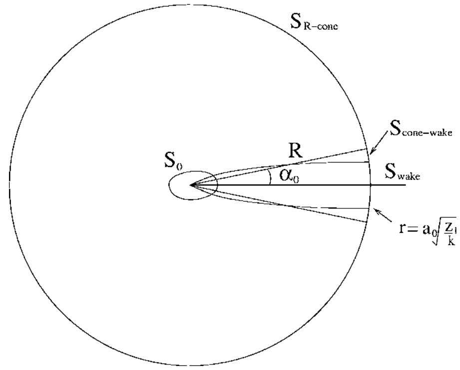

FIG. 2. The surface S_y.

3. and the surface $S_{R-\text{cone}}$ such that $|\mathbf{z}|=R$ and $\alpha>\alpha_0$,

where a_0 and α_0 are constants, and the Cartesian and spherical coordinate are such that $z_1=R\cos\alpha$, $z_2=R\sin\alpha\cos\theta$, $z_3=R\sin\alpha\sin\theta$. The approximations applied to the fundamental solutions within these three regions is given next.

The surface S_R is divided into the three areas $S_{R-\text{cone}}$, $S_{\text{cone-wake}}$, and S_{wake}, such that the following approximations are made in each area.

Area $S_{R-\text{cone}}$**:** Within this area $\alpha>\alpha_0$ and so the approximation

$$\frac{1}{R-z_1}<\frac{b_0}{R},\quad b_0=\frac{1}{1-\cos\alpha_0}\tag{10}$$

holds.

Area $S_{\text{cone-wake}}$**:** In this region $r/z_1\le\alpha_0$ and so we can apply the approximation

$$R-z_1=z_1\left\{1+\frac{r^2}{z_1^2}\right\}^{1/2}-z_1=\frac{r^2}{2z_1}-\frac{r^4}{8z_1^3}+O(r^6/z_1^5),\tag{11}$$

where O means "of order of." So,

$$\begin{aligned}e^{-k(R-z_1)}&=e^{-(kr^2/2z_1)(1+O(r^2/z_1^2))}\\&=e^{-kr^2/2z_1}(1+O(r^2/z_1^2))^{-kr^2/2z_1}\\&=e^{-kr^2/2z_1}(1+o(r^2/z_1^2))\end{aligned}\tag{12}$$

where o means "of order less than," since $(1+a)^{b+1}>1$ and so $(1+a)^{-b}<1+a$ for $a>0$, $b>0$. Finally, an element of area Δs over the surface is approximated by

$$\Delta s=R^2\sin\alpha\Delta\alpha\Delta\theta=r\Delta r\Delta\theta(1+O(r^2/z_1^2)).\tag{13}$$

Area S_{wake}**:** In this region $kr^2/2z_1\le a_0^2/2\ll1$, and so
AQ: #1 from Ref. 10, p. 69, Sec. 4.2.1,

$$\begin{aligned}e^{-k(R-z_1)}&=1-k(R-z_1)+\frac{k^2(R-z_1)^2}{2!}+O([R-z_1]^3)\\&=1-\frac{kr^2}{2z_1}+\frac{k^2r^4}{8z_1^2}+O(r^6/z_1^3),\end{aligned}\tag{14}$$

since

$$O\left(\frac{r^2}{z_1^2}\right)\ll O\left(\frac{r^2}{z_1}\right)$$

in the far-field region S_{wake} as $z_1\to\infty$.

The integral calculation of (9) is now evaluated over the three regions of the integral surface for the varying index values $1\le i,m\le3$. However, since $u_i^{(m)}=u_m^{(i)}$, $u_2^{(1)}$ has similar form to $u_3^{(1)}$, and $u_2^{(2)}$ has similar form to $u_3^{(3)}$, then it is sufficient to consider the four permutations $(i,m)=(2,3)$, $(2,2)$, $(1,2)$, and $(1,1)$.

Permutation $(i,m)=(2,3)$**:** Over the area $S_{R-\text{cone}}$, applying the approximation (10) to the Oseenlet given by (6) in the region $S_{R-\text{cone}}$ gives

$$\left|\frac{\partial\phi^{(2)}}{\partial z_3}\right|<\frac{b_0(1+b_0)}{4\pi\rho UR^2}\tag{15}$$

and so

$$\lim_{R\to\infty}\int\int_{S_{R-\text{cone}}}\left|\frac{\partial\phi^{(2)}}{\partial z_3}\right|\mathrm{d}s<\frac{b_0(1+b_0)}{\rho U}.\tag{16}$$

Similarly

$$\left|\frac{\partial\chi^{(2)}}{\partial z_3}\right|<\frac{b_0(1+kR+b_0)}{4\pi\rho UR^2}e^{-kR/b_0}\tag{17}$$

and so

$$\lim_{R\to\infty}\int\int_{S_{R-\text{cone}}}\left|\frac{\partial\chi^{(2)}}{\partial z_3}\right|\mathrm{d}s=0\tag{18}$$

and so

$$\lim_{R\to\infty}|u_j(\mathbf{y})|_{\max}\int\int_{S_{R-\text{cone}}}|u_3^{(2)}(\mathbf{z})|\mathrm{d}s=0\tag{19}$$

since $|u_j(\mathbf{y})|_{\max}\to0$ as $R\to\infty$.

Over the area $S_{\text{cone-wake}}$, applying the approximation (11) to the Oseenlet given by (6) in the region $S_{\text{cone-wake}}$ gives

$$\left|\frac{\partial\phi^{(2)}}{\partial z_3}\right|<\frac{1}{\pi\rho Ur^2}\tag{20}$$

and so using the approximation for elements of the surface (13) gives

$$\begin{aligned}\lim_{R\to\infty}\int\int_{S_{\text{cone-wake}}}\left|\frac{\partial\phi^{(2)}}{\partial z_3}\right|\mathrm{d}s&<\int_0^{2\pi}\int_{a_0\sqrt{z_1/k}}^{\alpha_0 z_1}\frac{a_2}{\pi\rho Ur}\mathrm{d}r\mathrm{d}\theta\\&=\frac{2}{\rho U}\{\ln(\alpha_0 z_1)\\&\quad-\ln(a_0\sqrt{z_1}/\sqrt{k})\}\end{aligned}\tag{21}$$

for some constant a_2. We note that if this integration was continued into the wake then the right-hand side of (21) would approach infinity and no bound would be obtained, which demonstrates the necessity for dividing the surface of the sphere up such that there is a wake region. Similarly

$$\left|\frac{\partial \chi^{(2)}}{\partial z_3}\right| < \frac{a_3}{z_1} e^{-kr^2/2z_1} \tag{22}$$

for some a_3 independent of the coordinate variables, since $1/r^2 \leq a_0^2/z_1$. So using the approximation for elements of the surface (13) gives

$$\lim_{R\to\infty} \int\int_{S_{\text{cone-wake}}} \left|\frac{\partial \chi^{(2)}}{\partial z_3}\right| \mathrm{d}s < \int_0^{2\pi} \int_{a_0\sqrt{z_1/k}}^{\alpha_0 z_1} \frac{a_3}{z_1} e^{-kr^2/2z_1} r \mathrm{d}r \mathrm{d}\theta$$

$$= \frac{2\pi a_3}{k} e^{-ka_0^2/2}, \tag{23}$$

which is bounded. In the far field, we expect the fluid velocity $u_j(\mathbf{y})$ to behave as a combination of the fundamental solutions $u_j^{(m)}(\mathbf{y})$ to leading order. So we expect that $|u_j(\mathbf{y})|_{\max} \to 0$ faster than $1/\ln R$ as $R \to \infty$. This means that combining the results (21) and (23) we expect

$$\lim_{R\to\infty} |u_j(\mathbf{y})|_{\max} \int\int_{S_{\text{cone-wake}}} |u_3^{(2)}(\mathbf{z})| \mathrm{d}s = 0. \tag{24}$$

Over the area S_{wake}, making use of the approximation (11), gives an approximation for $\phi^{(2)}$ in this region

$$\phi^{(2)} = \frac{1}{4\pi\rho U R} \frac{z_2}{R - z_1} = \frac{z_2}{2\pi\rho U r^2}\left(1 + \frac{r^2}{2z_1^2}\right)^{-1}$$

$$\times \left(1 - \frac{r^2}{4z_1^2}\right)^{-1} (1 + O(r^4/z_1^4))$$

$$= \frac{z_2}{2\pi\rho U r^2}\left(1 - \frac{r^2}{4z_1^2}\right)(1 + O(r^4/z_1^4)). \tag{25}$$

Further, making use of the approximation (14) for $e^{-k(R-z_1)}$ in this region then gives

$$\phi^{(2)} + \chi^{(2)} = \frac{z_2}{2\pi\rho U r^2}\left\{\frac{kr^2}{2z_1} - \frac{k^2 r^4}{8z_1^2} + O(r^6/z_1^3)\right\}, \tag{26}$$

so

$$\frac{\partial}{\partial z_3}(\phi^{(2)} + \chi^{(2)}) = -\frac{k^2 z_2 z_3}{8\pi\rho U z_1^2}(1 + O(r^2/z_1)). \tag{27}$$

Therefore

$$\lim_{R\to\infty} \int\int_{S_{\text{wake}}} |u_3^{(2)}(\mathbf{z})| \mathrm{d}s$$

$$= \lim_{R\to\infty} \frac{k^2}{4\rho U z_1^2} \int_0^{a_0\sqrt{z_1}} r^3 \mathrm{d}r (1 + O(r^2/z_1))$$

$$= \frac{k^2 a_0^4}{16\rho U}(1 + O(a_0^2)). \tag{28}$$

Combining all results together over the three surfaces $S_{R\text{-cone}}$, $S_{\text{cone-wake}}$, and S_{wake} on the surface of the sphere S_R then gives

$$\lim_{R\to\infty} |u_j(\mathbf{y})|_{\max} \int\int_{S_R} |u_3^{(2)}(\mathbf{z})| \mathrm{d}s = 0 \tag{29}$$

as expected.

Permutation $(i,m) = (2,2)$**:** Over the area $S_{R\text{-cone}}$, following the same approximations as for the permutation $(i,m) = (2,3)$, then in this region we have

$$\left|\frac{\partial \phi^{(2)}}{\partial z_2}\right| \leq \frac{a_4}{R^2}, \quad \left|\frac{\partial \chi^{(2)}}{\partial z_2}\right| \leq \frac{a_5}{R} e^{-kR/a_0}, \quad |\chi^*| \leq \frac{a_6}{R} e^{-kR/a_0} \tag{30}$$

for some constants a_4, a_5, and a_6. So, using the same argument as for the permutation $(i,m) = (2,3)$, then in this region we have

$$\lim_{R\to\infty} |u_j(\mathbf{y})|_{\max} \int\int_{S_{R\text{-cone}}} |u_2^{(2)}(\mathbf{z})| \mathrm{d}s = 0 \tag{31}$$

since $|u_j(\mathbf{y})|_{\max} \to 0$ as $R \to \infty$.

Over the area $S_{\text{cone-wake}}$, following the same approximations as for the permutation $(i,m) = (2,3)$, then in this region we have

$$\left|\frac{\partial \phi^{(2)}}{\partial z_2}\right| \leq \frac{a_7}{r^2}, \quad \left|\frac{\partial \chi^{(2)}}{\partial z_2}\right| \leq \frac{a_8}{z_1} e^{-kr^2/2z_1},$$

$$|\chi^*| \leq \frac{a_9}{z_1} e^{-kr^2/2z_1} \tag{32}$$

for some constants a_7, a_8, and a_9. So, using the same argument as for the permutation $(i,m) = (2,3)$, then in this area we have

$$\lim_{R\to\infty} |u_j(\mathbf{y})|_{\max} \int\int_{S_{R\text{-cone}}} |u_2^{(2)}(\mathbf{z})| \mathrm{d}s = 0 \tag{33}$$

since $|u_j(\mathbf{y})|_{\max} \to 0$ faster than $1/\ln R$ as $R \to \infty$.

Over the area S_{wake}, making use of the approximations (11) for $\phi^{(2)}$ and the approximation (14) for $e^{-k(R-z_1)}$ in this region gives

$$\frac{\partial}{\partial z_2}(\phi^{(2)} + \chi^{(2)}) = \frac{\partial}{\partial z_2}\left\{\frac{kz_2}{4\pi\rho U z_1}(1 + O(r^2/z_1))\right\}$$

$$= \frac{k}{4\pi\rho U z_1}(1 + O(r^2/z_1)). \tag{34}$$

Therefore

$$\lim_{R\to\infty} \int\int_{S_{\text{wake}}} |u_2^{(2)}(\mathbf{z})| \mathrm{d}s = \frac{ka_0^2}{4\rho U}(1 + O(a_0^2)), \tag{35}$$

which is bounded. Also in this region, $|\chi^*| \leq a_{10}/z_1$ and so combining all results together over the three surfaces $S_{R\text{-cone}}$, $S_{\text{cone-wake}}$, and S_{wake} on the surface of the sphere S_R then gives

$$\lim_{R\to\infty} |u_j(\mathbf{y})|_{\max} \int\int_{S_R} |u_2^{(2)}(\mathbf{z})| \mathrm{d}s = 0 \tag{36}$$

as expected.

Permutations $(i,m)=(1,1)$ and $(i,m)=(1,3)$**:** The analysis for these permutations give similar bounds, with the added simplification that $\frac{\partial}{\partial z_1}\ln(R-z_1)=-1/R$. This means that the condition (9) given by

$$\lim_{R\to\infty}\left\{|u_j(\mathbf{y})|_{\max}\int\int_{S_R}|u_i^{(m)}(\mathbf{y})|\mathrm{d}s\right\}=0_{ij}^{(m)} \qquad (37)$$

holds for all i,j, and m. So the evaluation of the far-field integral in the Green's function representation for steady Oseen flow is zero as expected.

ACKNOWLEDGMENT

This work was carried out during an EPSRC sponsored studentship for a doctorate in applied mathematics at the University of Salford.

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